Lec 10: 09/16/2009

Problems in One Dimension:

Free Particle:

The Hamiltonian for a free particle is $H = \frac{P^2}{2m}$. The eigenvalue problem is:

$H|E\rangle = E|E\rangle$

Working in the $|x\rangle$ basis, we have:

$$\langle x|H|E\rangle = \langle x|E|E\rangle \Rightarrow -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi_E(x) = E\psi_E(x) \quad -\infty < x < \infty$$

The solutions are:

$$\psi_E(x) \propto e^{\pm \frac{Px}{\hbar}} \quad P = \sqrt{2mE} \quad E < 0 \text{ (not acceptable, } \psi_E \text{ blows up at } \pm\infty)$$

$$\psi_E(x) \propto e^{\pm \frac{iPx}{\hbar}} \quad P = \sqrt{2mE} \quad E > 0$$

The normalized (to the Dirac delta function) solution is:

$$\psi_E(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{iPx}{\hbar}} \quad P = \pm\sqrt{2mE} \quad \text{(continuous spectrum)}$$

$$-\infty < P < +\infty$$

We note the two fold degeneracy; for any value of energy $E > 0$, there are two eigenstates.

The degeneracy happens when there is a continuous symmmetry under which the Hamiltonian is invariant. In the case of a free particle in one dimension translational invariance is the symmetry.

Also note that the degeneracy is broken by using an operator that commutes with H. Here we use the momentum operator P:

$$[H, P] = 0$$

$$P \psi_p(x) = p \psi_p(x) \qquad \psi_{p(x)} = e^{\frac{ipx}{\hbar}}$$

$$|E, +\rangle = |p = \sqrt{2mE}\rangle \qquad |E, -\rangle = |p = -\sqrt{2mE}\rangle$$

In general, once we find the energy eigenstates, we find the propagator U according to:

$$U_{(t)} = \sum_E |E\rangle\langle E| \, e^{-i\frac{Et}{\hbar}}$$

For a free particle:

$$U_{(t)} = \int_{-\infty}^{+\infty} |p\rangle\langle p| \, e^{-i\frac{p^2 t}{2m\hbar}} \, dp$$

From Schrodinger's equation:

$$|\Psi_{(t)}\rangle = U_{(t)} |\Psi_{(0)}\rangle \Rightarrow \langle x|\Psi_{(t)}\rangle = \langle x| U_{(t)} |\Psi_{(0)}\rangle$$

$$= \int_{-\infty}^{+\infty} \langle x| U_{(t)} |x'\rangle \langle x'|\Psi_{(0)}\rangle \, dx' \Rightarrow$$

$$\Psi_{(x,t)} = \int_{-\infty}^{+\infty} U_{(x,t;x')} \, \Psi_{(x',0)} \, dx'$$

Where:

$$U_{(x,t;x')} \equiv \langle x| U_{(t)} |x'\rangle$$

is matrix representation of the propagator in the X eigenbasis.

In general, we can choose the initial time to be t' instead of 0. Then:

$$\Psi(x,t) = \int_{-\infty}^{+\infty} U(x,t;x',t')\,\Psi(x',t')\,dx'$$

For a time-independent Hamiltonian, we have time translation invariance. This implies that $U(x,t;x',t')$ can only depend on $t-t'$, i.e. $U(x,t-t';x')$.

For a free particle we have:

$$U(x,t;x') \equiv \langle x|U(t)|x'\rangle = \int_{-\infty}^{+\infty} \langle x|p\rangle\langle p|x'\rangle\, e^{\frac{-ip^2t}{2m\hbar}}\,dp$$

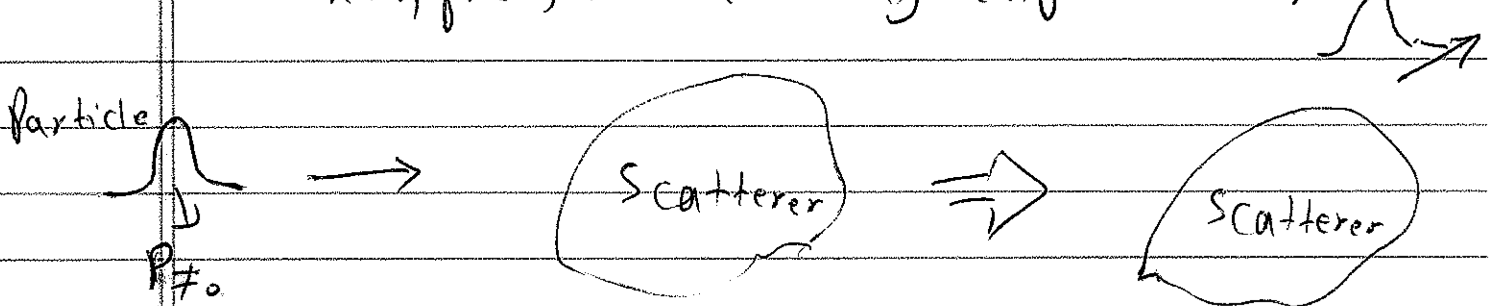
$$= \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} e^{\frac{ip(x-x')}{\hbar}}\, e^{-\frac{ip^2t}{2m\hbar}}\,dp = \left(\frac{m}{2\pi i\hbar t}\right)^{\frac{1}{2}} e^{\frac{im(x-x')^2}{2\hbar t}}$$

For an energy eigenstate we do not need to use $U(x,t;x')$ to find the time evolution, because it is simply a phase multiplication. However, for a superposition of energy eigenstates, we need to use $U(x,t;x')$.

We note that the probability to find a free particle that is in an energy eigenstate at a given point is constant:

$$|\psi_E(x)|^2 = \text{Const.}$$

Therefore it is equally probable to find it at any point $-\infty < x < +\infty$. But in practice, when we do experiments, a free particle is prepared in a state that has finite spatial extension. For example, in scattering experiments:



Such a localized state is called a wavepacket.

Gaussian Wavepacket:

An important example of a wavepacket is the Gaussian wavepacket. Consider as the initial stat the following wavepacket:

$$\Psi_{(x',0)} = e^{\frac{iP_0 x'}{\hbar}} \frac{e^{-\frac{x^2}{2\Delta^2}}}{(\pi\Delta^2)^{1/4}}$$

We have:

$$\langle X \rangle_{(0)} = 0 \qquad \Delta X_{(0)} = \frac{\Delta}{\sqrt{2}}$$

We can also find $\langle P \rangle_0$ and $\Delta P_{(0)}$. To do that, it will be easier to go to the eigenbasis of P:

$$\langle P | \Psi_{(0)} \rangle \equiv \Psi_{(P,0)} = \frac{1}{\pi^{1/4}} \sqrt{\frac{2\Delta}{\hbar}} \, e^{-\frac{2\Delta^2 (P-P_0)^2}{\hbar^2}}$$

$$\langle P \rangle_{(0)} = P_0 \qquad \Delta P_{(0)} = \frac{\hbar}{\sqrt{2}\,\Delta}$$

Note that $\Delta X_{(0)} \, \Delta P_{(0)} = \frac{\hbar}{2}$. As we will see later on, according to the Heisenberg uncertainty princible $\Delta X \, \Delta P \geq \frac{\hbar}{2}$.

Now, lets find $\Psi_{(x,t)}$:

$$\Psi_{(x,t)} = \int_{-\infty}^{+\infty} U_{(x,t;x')} \, \Psi_{(x',0)} \, dx'$$

Using the results for Gaussian integrals:

$$\psi(x,t) = \exp\left[\frac{ip_0}{\hbar}\left(x - \frac{p_0 t}{2m}\right)\right] \frac{e^{-\frac{x^2}{2\Delta^2(t)}}}{(\pi \Delta^2_{(t)})^{\frac{1}{4}}}$$

where;

$$\Delta_{(t)} = \Delta \left(1 + \frac{\hbar^2 t^2}{m^2 \Delta^4}\right)^{\frac{1}{2}}$$

Therefore;

$$\langle X \rangle_{(t)} = \frac{p_0 t}{m} \qquad \Delta X_{(t)} = \frac{\Delta}{\sqrt{2}} \left(1 + \frac{\hbar^2 t^2}{m^2 \Delta^4}\right)^2$$

This implies that the wavepacket moves at a velocity of $\frac{p_0}{m}$, and spreads as time goes on.

To find $\langle P \rangle_{(t)}$ and $\Delta P_{(t)}$, it is more Convenient to go the P eigenbasis;

$$\psi(p,t) = \int_{-\infty}^{+\infty} U(p,t;p',0)\, \psi(p',0)\, dp'$$

$$U(p,t;p',0) \equiv \langle p | U(t) | p' \rangle = \delta(p-p')\, e^{-\frac{ip'^2 t}{2m\hbar}}$$

$$\Rightarrow \psi(p,t) = \int_{-\infty}^{+\infty} \delta(p-p')\, e^{-\frac{ip'^2 t}{2m\hbar}}\, \psi(p',0)\, dp'$$

$$= \frac{1}{\pi^{\frac{1}{4}}} \sqrt{\frac{2\Delta}{\hbar}} \; e^{\frac{-ip^2 t}{2m\hbar}} \; e^{\frac{-2\Delta^2 (p-p_0)^2}{\hbar^2}}$$

Thus:

$$\langle P \rangle_{(t)} = P_0 \qquad \Delta P_{(t)} = \frac{\hbar}{\sqrt{2}\,\Delta}$$

This implies that $\langle P \rangle$ and ΔP are constant. This is expected since $[P, H] = 0$.

Note that $\Delta X_{(t)} \Delta P_{(t)} \geq \frac{\hbar}{2}$, in accordanced with the Heisenberg uncertainty principle, and the inequality is saturated at $t=0$.

Example: Why macroscopic particles are classical?
Consider a particle with mass $m = 100$ g. It is prepared in a Gaussian wavepacket with $\Delta = 10^{-5}$ mm. Let's say the experimental uncertainty in determining the position of the particle is 10^{-2} mm. Therefore the quantum uncertainty is not important

initially.

Note that $\Delta P = \frac{\hbar}{\sqrt{2}\Delta} \sim 10^{-26}\ kg\frac{m}{s}$ is constant in time

Also:

$$\Delta X_{(t)} = \frac{\Delta}{\sqrt{2}}\left(1 + \frac{\hbar^2 t^2}{m^2\Delta^4}\right)^{\frac{1}{2}}$$

The quantum uncertainty doubles when:

$$\frac{\hbar^2 t^2}{m^2\Delta^4} = 3 \Rightarrow t = \sqrt{3}\,\frac{m\Delta^2}{\hbar} \boxed{\sim 10^{17}\ sec}$$

But this is the age of the universe! Therefore the quantum uncertainty in the position (and momentum) remain much smaller than the experimental uncertainty at all times. The particle behaves "clasically".

Whenever the "experimental uncertainty" due to the limited accuracy of measurement is larger than the "intrinsic uncertainty" from quantum physics, we arrive at the classical limit.

This also tells us why quantum mechanics is relevant for microscopic systems, when the size is comparable with quantum uncertainty.